一、計算題

(每題 20 分共 100 分)

1. Let A be a nondefective $n \times n$ matrix with diagonalizing matrix X. Show that the matrix $Y = (X^{-1})^T$ diagonalizes A^T.

2. Let $L: R^3 \to R^3$ be the linear operator defined by $L(x) = A \cdot x$, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \text{ and let } v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Find the transition matrix V corresponding to a change of basis from $[v_1, v_2, v_3]$ to the standard basis $[e_1, e_2, e_3]$ and use it to determine the matrix B representing L with respect to $[v_1, v_2, v_3]$.

3. Find the eigenvalues and the corresponding eigenspaces for the following matrix:

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

4. Factor the following matrix A into a product $A = XDX^{-1}$, where D is a diagonal matrix.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}$$

5. Consider the following system of equations $A \cdot x = b$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 \\ -2 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

a. Determine its solution set by Gaussian elimination.
b. Is this system consistent?
c. For the row vectors of A, let

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}, x_5 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

Pare down the set $\{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5\}$ to form a basis for the subspace Span$\{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5\}$.